Properties of Functions

And again…

f(Linda) = Moscow	g(Linda) = Moscow
f(Max) = Boston	g(Max) = Boston
f(Kathy) = Hong Kong	g(Kathy) = Hong Kong
f(Peter) = Boston	g(Peter) = New York

Is f one-to-one?

No, Max and Peter are mapped onto the same element of the image.

Is g one-to-one?

Yes, each element is assigned a unique element of the image.

Properties of Functions

How can we prove that a function f is one-to-one?

Whenever you want to prove something, first take a look at the relevant definition(s):

$\forall x, y \in A\ (f(x) = f(y) \rightarrow x = y)$

Example:

$f: \mathbf{R} \rightarrow \mathbf{R}$

$f(x) = x^2$

Disproof by counterexample:

$f(3) = f(-3)$, but $3 \neq -3$, so f is not one-to-one.

Properties of Functions

… and yet another example:

$f: \mathbf{R} \to \mathbf{R}$

$f(x) = 3x$

One-to-one: $\forall x, y \in A \ (f(x) = f(y) \to x = y)$

To show: $f(x) \neq f(y)$ whenever $x \neq y$ (indirect proof)

$x \neq y$

$\Leftrightarrow 3x \neq 3y$

$\Leftrightarrow f(x) \neq f(y)$,

so if $x \neq y$, then $f(x) \neq f(y)$, that is, f is one-to-one.

Properties of Functions

A function $f:A \rightarrow B$ with $A,B \subseteq R$ is called strictly increasing, if

$\forall x,y \in A \ (x < y \rightarrow f(x) < f(y))$,

and strictly decreasing, if

$\forall x,y \in A \ (x < y \rightarrow f(x) > f(y))$.

Obviously, a function that is either strictly increasing or strictly decreasing is one-to-one.

Properties of Functions

A function $f:A\to B$ is called onto, or surjective, if and only if for every element $b\in B$ there is an element $a\in A$ with $f(a) = b$.

In other words, f is onto if and only if its range is its entire codomain.

A function $f: A\to B$ is a one-to-one correspondence, or a bijection, if and only if it is both one-to-one and onto.

Obviously, if f is a bijection and A and B are finite sets, then $|A| = |B|$.

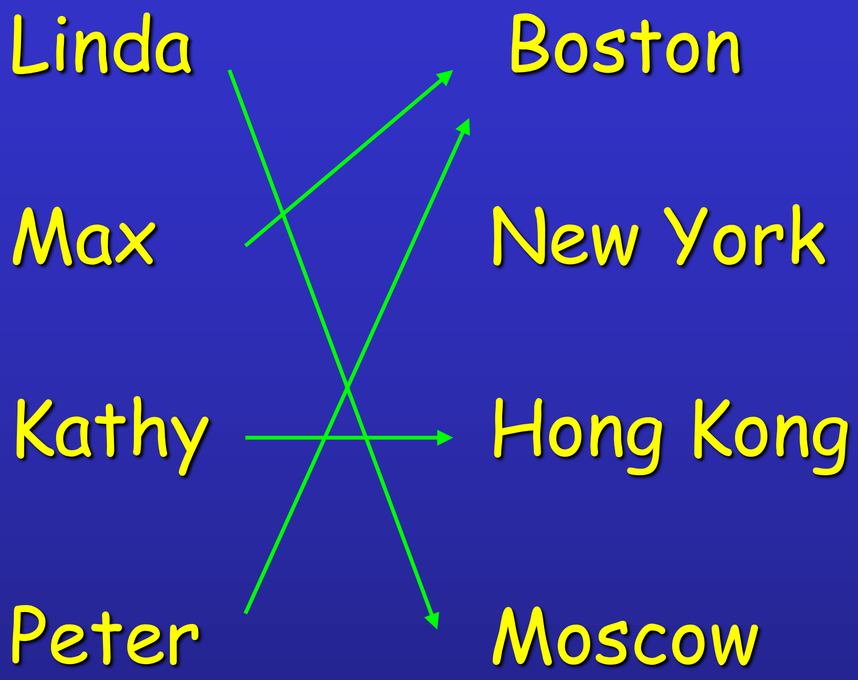
Properties of Functions

Examples:

In the following examples, we use the arrow representation to illustrate functions $f:A \to B$.

In each example, the complete sets A and B are shown.

Properties of Functions



Is f injective?

No.

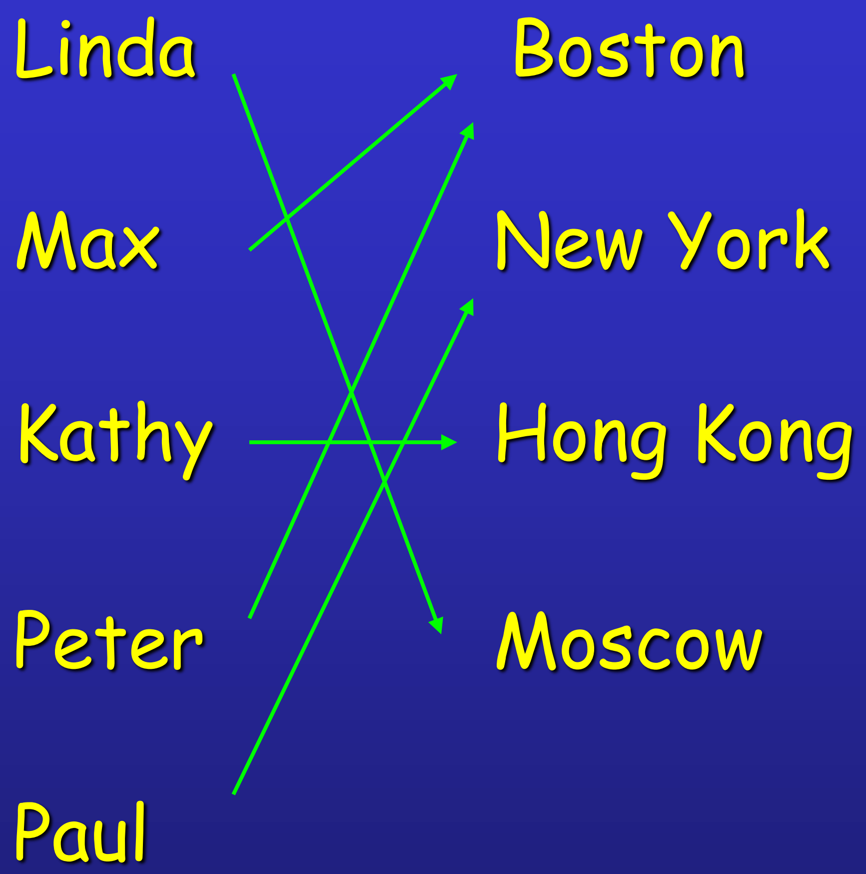
Is f surjective?

No.

Is f bijective?

No.

Properties of Functions



Is f injective?

No.

Is f surjective?

Yes.

Is f bijective?

No.

Properties of Functions

Is f injective?

Yes.

Is f surjective?

No.

Is f bijective?

No.

Properties of Functions

Is f injective?

No! f is not even a function!

Properties of Functions

Is f injective?

Yes.

Is f surjective?

Yes.

Is f bijective?

Yes.

Inversion

An interesting property of bijections is that they have an **inverse function**.

The **inverse function** of the bijection $f:A \to B$ is the function $f^{-1}:B \to A$ with

$f^{-1}(b) = a$ whenever $f(a) = b$.

Inversion

Example:

$f(\text{Linda}) = \text{Moscow}$
$f(\text{Max}) = \text{Boston}$
$f(\text{Kathy}) = \text{Hong Kong}$
$f(\text{Peter}) = \text{Lübeck}$
$f(\text{Helena}) = \text{New York}$

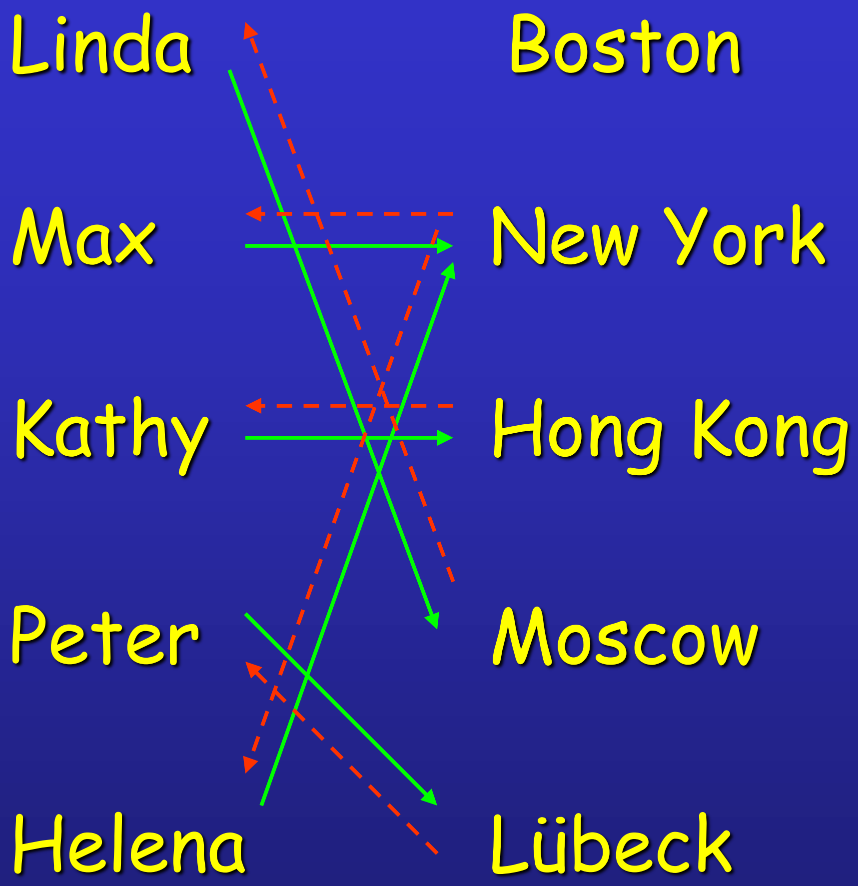
Clearly, f is bijective.

The inverse function f^{-1} is given by:

$f^{-1}(\text{Moscow}) = \text{Linda}$
$f^{-1}(\text{Boston}) = \text{Max}$
$f^{-1}(\text{Hong Kong}) = \text{Kathy}$
$f^{-1}(\text{Lübeck}) = \text{Peter}$
$f^{-1}(\text{New York}) = \text{Helena}$

Inversion is only possible for bijections (= invertible functions)

Inversion



$f^{-1}: C \to P$ is no function, because it is not defined for all elements of C and assigns two images to the pre-image New York.

Composition

The **composition** of two functions $g:A \rightarrow B$ and $f:B \rightarrow C$, denoted by $f \circ g$, is defined by

$(f \circ g)(a) = f(g(a))$

This means that

- **first**, function g is applied to element $a \in A$, mapping it onto an element of B,
- **then**, function f is applied to this element of B, mapping it onto an element of C.
- **Therefore**, the composite function maps from A to C.

Composition

Example:

$f(x) = 7x - 4$, $g(x) = 3x$,

$f:\mathbf{R}\rightarrow\mathbf{R}$, $g:\mathbf{R}\rightarrow\mathbf{R}$

$(f \circ g)(5) = f(g(5)) = f(15) = 105 - 4 = 101$

$(f \circ g)(x) = f(g(x)) = f(3x) = 21x - 4$